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On the mathematics of fluidization

Part 1. Fundamental equations and wave propagation

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When the upward flow of a fluid through a bed of particles of appropriate and almost uniform size is rapid enough so that the drag on each particle is as great as the particles buoyant weight, the particles do not remain close packed and the bed is said to be fluidized. Industrial uses of fluidized beds in the chemical and petroleum industries in particular are already extensive. Uses in the atomic-energy industry are being developed.

In this paper a mathematical model which describes the phenomena on a continuum basis is deduced. With this model we find that the system is unstable to small internal disturbances. Alternatively, we find that surface waves can be propagated (with attenuation) in the composite fluid and these waves for fluidized beds with a high ratio of solids density to fluid density are stable. These results are in agreement with experiment. Hot beds, where strongly exothermic reactions may be taking place, centrifugal beds (beds fluidized within a rotating system), and electromagnetic beds (those in which the particulate phase is electrically conducting) are all shown to be unstable to small internal disturbances.

The equations derived here may also be used as approximate equations for dispersed particle two-phase flow.

1. Introduction

Fluidization is the process whereby a bed of particles which, in the laboratory, range in size from a few mm to about 10^{-3} cm in diameter, is subjected to an upward flow of fluid through the bed at such a speed that the gravitational forces on the particles are balanced by the drag forces. As the fluid flow increases from zero the particles experience a drag which eventually becomes large enough to make them move. The particles will remain in contact as the flow increases, until the drag force equals the gravitational force. With increase in flow beyond this stage the particles move away from each other so as to keep the drag force equal to the gravitational force, resulting in some expansion of the bed. The bed in this state is said to be fluidized. At considerably higher fluid velocities the solids in the bed may be transported with the fluid. This latter phenomenon is sometimes called pneumatic transport.

Widespread use of fluidized beds in industry has been accepted in the last 20 years, particularly in the petroleum industry. For example, the catalytic cracking of petroleum vapour and the catalytic regeneration can be made continuous by the use of fluidized beds. In the atomic-energy industry potential uses include the calcination of waste fission products and heat exchangers in nuclear reactors. Ore roasting, the carbonization of coal, and the manufacture of fairly pure tungsten are all examples of successfully applied fluidization techniques.

Although there is a very large literature on fluidization (general references are Botterill 1958, Franz 1962, Leva 1959, Zenz & Othmer 1960, the 1961 Symposium on Fluidization and the 1962 Symposium on Fluid/Particle Interactions), it appears that many of the fundamental principles are still not understood. For example, practically all gas fluidized beds are characterized by the appearance of bubbles of gas (or regions with practically no particles) moving up through the bed. The nature of their growth, shape and stability is not fully understood. Unlike an ordinary gas bubble in a liquid the fluid particles within the void do not remain there, but are continually changing. Such bubbles do not appear in most liquid fluidized beds. It appears empirically that bubbles will form in a fluidized bed if the ratio of the density of the solids to that of the fluid is greater than about 10. Most liquid fluidized beds have such a ratio less than 10. Water and lead shot is a borderline case and bubbles accordingly sometimes appear. When these bubbles reach the surface of the bed they generate surface waves, which from observations are damped very quickly. In fact, they are damped very much quicker than ordinary waves on the surface of a liquid. In beds where bubbles do not appear, surface waves generated at the surface are also quickly damped. In beds where the density of the two components are comparable this may or may not be the case.

Symposia in 1961 and 1962 on fluidization indicated the necessity of a more fundamental approach to the subject, and most of the papers were directed towards this.

A suitable system to study consists of non-porous spherical particles of equal size fluidized by air: particle diameters in the range 0·1–1 mm are most convenient. Glass ballotini or graded sand are good materials to use. For example, ballotini of 0·5 mm fluidizes at a superficial minimum velocity of approximately 25 cm/sec; this velocity varies roughly as the square of the particle diameter. Liquid fluidized beds naturally require considerably smaller fluid velocities. For systems which are typical the Reynolds number, based on the particle diameter and the superficial fluid velocity, is of the order of 10.

Several attempts have been made to set up equations for a fluidized or general two-phase system.

Carrier & Cashwell (1956) considered the dynamics of a fluidized bed in the case of large solids-to-gas density ratio in which large temperatures are generated. Both internal and surface waves were considered and both were found to be unstable. They suggest that the dissipative mechanism which might result in attenuation of surface waves is due to a form of the bulk viscosity. As shown below, inclusion of this in the momentum equation is necessary but the apparent (but incorrect) instability of the surface waves arose from the use of an inadequate treatment of the free-surface boundary conditions. In this analysis, we derive conservation equations analogous to theirs but including certain further terms, and using an *appropriate* boundary condition, including *both* the bulk and shear viscosities, we find that surface waves are stable.

Van Deemter & Van der Laan (1960) attempted to obtain momentum equations for dispersed two-phase flow but only in a formal way. The nature of terms like the stress tensors and drag forces are not discussed. As they stand the equations are not of practical use. Compressibility effects are excluded.

Hinze (1961) derived a set of equations for the momentum and energy balance of a flowing homogeneous suspension with slip between the two phases by considering the effect of the particles as external forces acting on the continuous fluid phase. The stress tensor he suggested involves a composite velocity made up of the solids velocity and fluid velocity. The effect of the particle assembly on the shear viscosity in this tensor is not discussed nor is the bulk viscosity included. As shown below (see the Appendix) the latter term is in general large compared with the shear viscosity term and is important dissipatively. His equations involve the solids velocity, the fluid velocity and a further velocity which represents the transport of the two phases together. The equations he derives for the total momentum of the two phases, together, involve a virtual-mass term: this violates Newton's third law. The stability of the system he considered is not discussed. Again compressibility effects are excluded.

The cases considered by Van Deemter & Van der Laan (1960) and Hinze (1961) are not strictly fluidized states unless appropriate conditions on the balance of forces is applied as a form of boundary condition. The form of the equations is not changed.

Jackson (1963) in Part I obtained equations for the momentum balance of a fluidized system. His equations are effectively those obtained by Carrier & Cashwell (1956) when temperature and compressibility effects (energy equations) are excluded from the latter except that Jackson included a very artificial additional-mass term, dependent only on the fluid velocity. Neither viscous stress terms nor compressibility effects are included. He studied small internal waves and found them to be unstable On the measure of the instability he found, he classified various fluidized systems. His equations, however, would also result in unstable surface waves. In view of the importance of the stress tensors on the growth of small disturbances and their stability or non-stability such a classification is premature.

In this paper a set of conservation, momentum and energy equations (including compressibility and heat effects) are obtained by considering the two phases on a continuum basis and expressing, in the usual way, the appropriate balance of momentum and energy passing through a surface in the composite fluid. A solids and fluid stress tensor are included and discussed (see § 2 and the Appendix). The equations here derived differ from all so far suggested. It is found that, consistent with observation, small internal waves are unstable and surface waves stable (in the case when the density ratio of solids to fluid is large). The method of treating the surface boundary conditions is given in § 5. The case of internal waves in fluidized beds (without compressibility and heat effects) is discussed in full in § 4. These beds will be referred to as incompressible beds. In § 6 the effects of large temperature changes within the bed are studied: such beds will be referred to as hot beds. What will be described as centrifugal beds are briefly discussed in § 7 together with the effect on the equations of such a rotating system. If the particles are of an electrically conducting material applied electromagnetic fields will affect the motion of the particles and hence of the fluid. The effect on bubble motion could be considerable. The stability of a bed with a magnetic field aligned with the gravitational force is briefly discussed in the case of an infinitely electrically conducting particle medium in § 8. The experimental evidence available (including studies of gas fluidized, liquid fluidized and hot beds) on stability and in particular on surface wave propagation bears out the analytical results derived in this paper.

2. Conservation and momentum equations

The equations are derived on the basis that the bed, uniformly fluidized by a fluid of density ρ_f,‡ consists of particles of identical mass m, radius a, volume τ and density ρ_s (in general a constant). Let $\mathbf{v}_s$, $\mathbf{v}_f$ be the average velocities of the particles and fluid respectively over some volume large compared with the particle size. Both phases of the flow are thus treated as continuum flows; that is rapid local variations of the $\mathbf{v}_s$ and $\mathbf{v}_f$ and the fluid pressure p are averaged out. Let the number density of the particles be n.† Let $Z\,(=n\tau)$† be the fraction of the particles in a unit volume. Conservation equations for the particulate and fluid phase, respectively, are

$$\operatorname{div}\rho_s Z\mathbf{v}_s = -\,\partial(\rho_s Z)/\partial t, \tag{1}$$

$$\operatorname{div}\rho_f(1-Z)\mathbf{v}_f = -\,\partial\rho_f(1-Z)/\partial t. \tag{2}$$

The total momentum equation is obtained in the usual way by considering the momentum balance within a fixed surface S enclosing a volume V of the bed. Let V_s and V_f be the fractions of V occupied respectively by the solids and fluid, the volumes being enclosed by surfaces S_s and S_f. If we neglect boundary particles on S the momentum balance is

$$\frac{d}{dt_f}\int_{V_f}\rho_f\mathbf{v}_f\,dV_f+\frac{d}{dt_s}\int_{V_s}\rho_s\mathbf{v}_s\,dV_s+g\left[\int_{V_f}\rho_f\,dV_f+\int_{V_s}\rho_s\,dV_s\right]\mathbf{j}$$
$$-\int_{V_f}\rho_f\mathbf{F}_f\,dV_f-\int_{V_s}\rho_s\mathbf{F}_s\,dV_s-\int_{S_f}\boldsymbol{\pi}_f.\,d\mathbf{S}_f-\int_{S_s}\boldsymbol{\pi}_s.\,d\mathbf{S}_s=0, \tag{3}$$

where $\quad d/dt_f = \partial/\partial t+\mathbf{v}_f.\mathrm{grad}, \quad d/dt_s = \partial/\partial t+\mathbf{v}_s.\mathrm{grad},$

g is the gravitational constant, $\mathbf{j}$ a unit vertical vector, $\mathbf{F}_s$, $\mathbf{F}_f$ the external forces per unit mass on the solids and fluid respectively, and $\boldsymbol{\pi}_s$, $\boldsymbol{\pi}_f$ the viscous stress

† As an example, if air and 0·5 mm glass spheres (ballotini) are used, a typical bed in a fluidized state has $n \doteqdot 9000\ \mathrm{cm}^{-3}$, $\tau = 6{\cdot}55\times10^{-5}\ \mathrm{cm}^3$, $Z \doteqdot 0{\cdot}59$, and the interstitial velocity of the air is approximately 50 cm/sec.

‡ A partial list of symbols is given at the end of the paper.

tensors (which include the pressures) of the solids and fluid respectively. Since

$$dV_s = Z\,dV, \quad dV_f = (1-Z)\,dV,$$

and using the divergence on the last two integrals, equation (3) gives

$$\rho_f(1-Z)\,d\mathbf{v}_f/dt_f + \rho_s Z\,d\mathbf{v}_s/dt_s = -g[\rho_f(1-Z)+\rho_s Z]\,\mathbf{j} + \rho_f(1-Z)\,\mathbf{F}_f + \rho_s Z\mathbf{F}_s + (1-Z)\,\mathrm{div}\,\boldsymbol{\pi}_f + Z\,\mathrm{div}\,\boldsymbol{\pi}_s. \quad (4)$$

Note that the respective integrals over the actual particles themselves of $\boldsymbol{\pi}_f$ and $\boldsymbol{\pi}_s$ cancel as action and reaction. In (3) and (4) we have omitted the effect of electrostatic forces which are negligible except in a few extreme cases, as for example when very fine particles are fluidized with dry gas. Such effects near the containing walls are dominated by the boundary-layer effects. These cases and regions are excluded from this discussion.

We formally write

$$\pi_{f_{ij}} = -p\delta_{ij} + \sigma_{f_{ij}}, \quad \pi_{s_{ij}} = -p_s\delta_{ij} + \sigma_{s_{ij}}, \quad (5)$$

where p is the fluid pressure, and p_s the solids pressure (a form of collision pressure), and

$$\sigma_{f_{ij}} = \mu(\partial v_{f_i}/\partial x_j + \partial v_{f_j}/\partial x_i) + (\zeta - \tfrac{2}{3}\mu)\,\delta_{ij}\,\mathrm{div}\,\mathbf{v}_f, \quad (6)$$

$$\sigma_{s_{ij}} = \mu_s(\partial v_{s_i}/\partial x_j + \partial v_{s_j}/\partial x_i) + (\zeta_s - \tfrac{2}{3}\mu_s)\,\delta_{ij}\,\mathrm{div}\,\mathbf{v}_s, \quad (7)$$

where μ, μ_s are the shear viscosities and ζ, ζ_s the bulk viscosities of the fluid and solids respectively. Equations (4) and (5) formally give the usual momentum equations in the limiting cases $Z \to 0$ and $Z \to 1$. The latter case being that in which the particles become smaller but more numerous as $Z \to 1$ so that p_s does become the usual pressure. The former limit must be interpreted as that in which the particles become smaller and smaller.

The particle collision term (that is the p_s-term) is in general negligible and will be taken to be zero for the following reasons. It is an experimental fact that in fluidized beds practically no noise is noticeable, which would not be the case if collisions were frequent. There is very little attrition observed. The particle motion (both visually and by X-rays) around a bubble suggests that there is little interference between neighbouring particles. In fact there is no mechanism by which a solids pressure could be transmitted except by frequent collisions, which is not the case, or by high-frequency oscillations of the particles which are inappropriate because of the high energy which would be required and dissipated. Thus, in (5) we shall take p_s to be zero. To be consistent we must replace $(1-Z)$ by unity in the p-term in (4).

The nature and form of μ_s and ζ_s depends on the viscous effects of neighbouring particles in the flow field. In the Appendix it is shown that reasonable forms for μ_s and ζ_s are†

$$\mu_s = \mu A D_s, \quad \zeta_s = \mu B D_s^3,$$

where A and B are constants of $O(1)$ and $D_s = Z/(Z_s - Z)$, where Z_s is the saturation value of Z (that is when the particles are not fluidized) for given particles.

† The actual form of μ_s and ζ_s are not specifically required in this paper, but their inclusion is required in the equations.

D_s is of $O(a/h)$† where $2h$ is the distance between the particles. In normal fluidized beds $D_s \gg 1$. Thus a reasonable approximate form for (7) is

$$\sigma_{s_{ij}} = \mu A D_s(\partial v_{s_i}/\partial x_j + \partial v_{s_j}/\partial x_i) + \mu D_s(BD_s^2 - \tfrac{2}{3}A)\,\delta_{ij}\,\mathrm{div}\,\mathbf{v}_s. \tag{8}$$

The total momentum equation (4) is thus approximated by

$$\rho_f(1-Z)\,d\mathbf{v}_f/dt_f + \rho_s Z\,d\mathbf{v}_s/dt_s = -g[\rho_f(1-Z)+\rho_s Z]\,\mathbf{j} - \mathrm{grad}\,p + \rho_f(1-Z)\,\mathbf{F}_f + \rho_s Z\mathbf{F}_s + (1-Z)\,\mathrm{div}\,\boldsymbol{\sigma}_f + Z\,\mathrm{div}\,\boldsymbol{\sigma}_s. \tag{9}$$

We now consider the solids momentum equation in a similar way to the above and get

$$\rho_s Z\,d\mathbf{v}_s/dt_s + g\rho_s Z\mathbf{j} - \rho_s Z\mathbf{F}_s - Z\,\mathrm{div}\,\boldsymbol{\sigma}_s = \text{interaction forces due to the fluid.}$$

(The fluid momentum equation is obtained by subtracting the above from (9).) The main interaction forces are (i) the buoyancy, (ii) an additional mass term, and (iii) the viscous drag.

The buoyancy term is $\rho_f Zg\mathbf{j}$. The additional mass term is taken as

$$C\rho_f Z\,d(\mathbf{v}_f - \mathbf{v}_s)/dt, \quad d/dt = \partial/\partial t + (\mathbf{v}_f - \mathbf{v}_s).\mathrm{grad}, \tag{10}$$

where C is an unknown function of the particle geometry and fluidized lattice; it is taken to be of $O(1)$. This is a generalization of the case of a single sphere in an unbounded inviscid flow, in which case $C = 0{\cdot}5$. The actual form of the acceleration term in (10) is conjecture. A term like $C\rho_f Z(d\mathbf{v}_f/dt_f - d\mathbf{v}_s/dt_s)$ is also reasonable. A comparable generalization of a form suggested by Corrsin & Lumley (1956) for the motion of a single particle in a turbulent fluid would result in a term $C\rho_f Z[d\mathbf{v}_f/dt_f + \tfrac{1}{2}d(\mathbf{v}_f - \mathbf{v}_s)/dt_s]$. At the present time, the form of such a term can only be suggested. In any case in gas-fluidized beds such a term is of second order in the gas-to-solids density ratio (when compared with other terms in the momentum equation involving ρ_s) and therefore is not of first importance in such beds. The effect on other beds can be estimated.

The Reynolds number Re, based on the particle diameter $2a$, is $2\rho_f a\,|\mathbf{v}_f - \mathbf{v}_s|/\mu$. In most fluidized beds Re is $O(10)$. The drag force, $\mathbf{F}$, on a particle will, in general, be a function of Z, $(\mathbf{v}_f - \mathbf{v}_s)$ and Re. Rowe (1961) has found, experimentally, a curve for the ratio of the forces on a particle, for a given velocity of the fluid, when in a uniform assembly of similar particles to when it is isolated. The form for $\mathbf{F}$ may be written as

$$|\mathbf{F}| = C_D(\pi a^2)\,\tfrac{1}{2}\rho_f(\mathbf{v}_f - \mathbf{v}_s)^2\,(1 + 0{\cdot}68a/h),\dagger \tag{11}$$

where h/a is the dimensionless spacing of the particles, $2h$ apart, referred to a as the basic length and C_D is the appropriate drag coefficient. In the range of Reynolds number relevant to most fluidized beds C_D is given empirically by

$$C_D = (24/\mathrm{Re})\,(1 + 0{\cdot}15\,\mathrm{Re}^{0{\cdot}657}).$$

The first term gives the Stokes drag which is usually sufficient for Re $< 0{\cdot}2$. From (11) the force $\mathbf{F}$ on a particle in a fluidized bed may be approximated by

$$\mathbf{F} = (1 + 0{\cdot}15\,\mathrm{Re}^{0{\cdot}657})\,6\pi\mu a(1 + GD_s)\,(\mathbf{v}_f - \mathbf{v}_s),$$

† The limit $h \to 0$ is not included: $h > 0$ always.

where G is a constant of $O(1)$. We anticipate that the $(1+0\cdot15\text{Re}^{0\cdot657})$ term in $\mathbf{F}$ does not vary greatly over the range of relative velocities found in fluidized beds, and accordingly shall take it to be effectively constant and put

$$H = \text{Average}\,[6\pi(1+0\cdot15\,\text{Re}^{0\cdot657})].$$

Thus for a given fluid particle size we shall consider $\mathbf{F}$ to be given by the product of the relative velocity times a function of Z alone. We therefore write the viscous drag force on the particulate flow as $\mathbf{D}$, where

$$\left.\begin{aligned}\mathbf{D} &= (\mu a H/m)\,\rho_s Z(1+GD_s)\,(\mathbf{v}_f-\mathbf{v}_s)\\ &= D(Z)\,(\mathbf{v}_f-\mathbf{v}_s),\\ \text{with}\qquad D(Z) &= (\mu a H/m)\,\rho_s Z(1+GD_s).\end{aligned}\right\} \tag{12}$$

Figure 1† illustrates the form of $D(Z)$, where Z_0 is the value at incipient fluidization. Note that $[\partial D(Z)/\partial Z]_{Z=Z_0}$ is large.

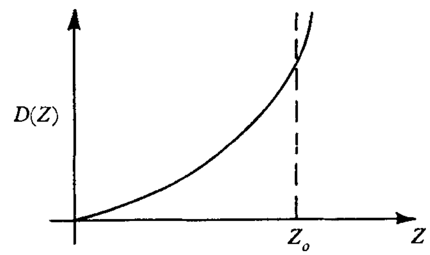

FIGURE 1. Drag force on the particulate flow as a function of particle density.

The momentum equation for the solids, using (10) and (12), is thus approximated by

$$\begin{aligned}\rho_s Z\, d\mathbf{v}_s/dt_s = &-gZ(\rho_s-\rho_f)\,\mathbf{j}+\rho_s Z\mathbf{F}_s+Z\,\text{div}\,\boldsymbol{\sigma}_s\\ &+(\mu a H/m)\,\rho_s Z(1+GD_s)\,(\mathbf{v}_f-\mathbf{v}_s)+C\rho_f Z\,d(\mathbf{v}_f-\mathbf{v}_s)/dt,\end{aligned} \tag{13}$$

and that for the fluid, on subtracting (13) from (9), by

$$\begin{aligned}\rho_f(1-Z)\,d\mathbf{v}_f/dt_f = &-g\rho_f\,\mathbf{j}+\rho_f(1-Z)\,\mathbf{F}_f-\text{grad}\,p+(1-Z)\,\text{div}\,\boldsymbol{\sigma}_f\\ &-(\mu a H/m)\,\rho_s Z(1+GD_s)\,(\mathbf{v}_f-\mathbf{v}_s)-C\rho_f Z\,d(\mathbf{v}_f-\mathbf{v}_s)/dt.\end{aligned} \tag{14}$$

It should be noted at this stage that equations (1), (2), (9), (13) and (14) are not necessarily restricted to fluidized beds but also apply to general dispersed two-phase flow. It is envisaged that the velocity of the fluid in these cases is greater than that necessary for incipient fluidization.

An immediate simplification of the equations is obtained if we restrict ourselves to gas fluidized beds or any in which $\rho_f/\rho_s \ll 1$ (in fact $< 0\cdot1$ is sufficient). In this case we may neglect all the ρ_f- compared with the ρ_s-terms. If there are no compressibility effects expected (as, for example, exists in hot beds, § 6) the velocities are such that $\rho_f \doteqdot$ const. ρ_s is also taken to be constant. The gas stress tensor is small compared with the solids stress tensor and may be neglected in comparison. If we further omit external forces (other than gravity) $\mathbf{F}_f = 0 = \mathbf{F}_s$.

† A similar curve has been obtained theoretically by Kuwabara (1959).

Thus, from equations (1), (2), (13) and (14), using the above approximations, we obtain the following set of equations which are a closed approximate set for most gas fluidized beds in which temperature and compressibility effects are excluded:

$$\left.\begin{aligned} \operatorname{div} Z\mathbf{v}_s &= -\partial Z/\partial t, \\ \operatorname{div}(1-Z)\mathbf{v}_f &= \partial Z/\partial t, \\ \rho_s Z\, d\mathbf{v}_s/dt_s &= -g\rho_s Z\mathbf{j} + Z \operatorname{div} \boldsymbol{\sigma}_s + D(Z)(\mathbf{v}_f - \mathbf{v}_s), \\ \operatorname{grad} p &= -D(Z)(\mathbf{v}_f - \mathbf{v}_s), \end{aligned}\right\} \tag{15}$$

where $D(Z)$ is given by (12) and $\boldsymbol{\sigma}_s$ by (8).

3. Energy and state equations

We now consider the conceptual surface S, above, to move with the fluid and consider in the usual way the rate at which work is done on the surface. This gives, for the solids and fluids together,

$$\begin{aligned} &\int_{S_f} (\mathbf{v}_f \cdot \boldsymbol{\pi}_f) \cdot d\mathbf{S}_f + \int_{S_s} (\mathbf{v}_s \cdot \boldsymbol{\pi}_s) \cdot d\mathbf{S}_s + \int_{V_f} \rho_f (\mathbf{F}_f \cdot \mathbf{v}_f)\, dV_f + \int_{V_s} \rho_s (\mathbf{F}_s \cdot \mathbf{v}_s)\, dV_s \\ &\qquad - \int_{V_f} g\rho_f \mathbf{j} \cdot \mathbf{v}_f\, dV_f - \int_{V_s} \rho_s g \mathbf{j} \cdot \mathbf{v}_s\, dV_s + \int_{S_f} \mathbf{q}_f \cdot d\mathbf{S}_f \\ &\qquad + \int_{S_s} \mathbf{q}_s \cdot d\mathbf{S}_s + \int_{V_f} \rho_f R_f\, dV_f + \int_{V} \rho_s R_s\, dV_s \\ &= \int_{V_f} \rho_f [\mathbf{v}_f \cdot d\mathbf{v}_f/dt_f + d(I_f - p/\rho_f)/dt_f]\, dV_f + \int_{V_s} \rho_s [\mathbf{v}_s \cdot d\mathbf{v}_s/dt_s + d(I_s - p_s/\rho_s)/dt_s]\, dV_s, \end{aligned} \tag{16}$$

where $\mathbf{q}_f$ and $\mathbf{q}_s$ are the heat flux vectors of the fluid and solids, R_f, R_s are the external or internal reaction heat inputs per unit mass to the fluid and solids, I_f, I_s are the fluid and solids enthalpy, equal to $c_p T_f$, $c_s T_s$ respectively, with c_p the specific heat at constant pressure of the fluid, c_s the specific heat of the solids, and T_f, T_s are the respective temperatures of the fluid and solids. Write

$$\left.\begin{aligned} (\boldsymbol{\pi}_f \cdot \operatorname{grad}) \cdot \mathbf{v}_f &= \Phi_f - p \operatorname{div} \mathbf{v}_f, & \Phi_f &= \sigma_{f_{ij}}\, \partial v_{f_j}/\partial x_i, \\ (\boldsymbol{\pi}_s \cdot \operatorname{grad}) \cdot \mathbf{v}_s &= \Phi_s - p_s \operatorname{div} \mathbf{v}_s, & \Phi_s &= \sigma_{s_{ij}}\, \partial v_{s_j}/\partial x_i, \end{aligned}\right\} \tag{17}$$

where Φ_f and Φ_s are the viscous dissipation functions defined by equations (17). In the usual way we now use the divergence theorem, the conservation equations (1) and (2) and the separate fluid and solids momentum equations (which add to give (4)) (that is, we are retaining p_s at this stage) on equation (16) which gives one form of the total energy equation as

$$\begin{aligned} &[\rho_f(1-Z)\, d(c_p T_f)/dt_f - dp(1-Z)/dt_f - (1-Z)\,\Phi_f - (1-Z) \operatorname{div} \mathbf{q}_f - \rho_f(1-Z) R_f] \\ &\qquad + [\rho_s Z\, d(c_s T_s)/dt_s - d(Zp_s)/dt_s - Z\Phi_s - Z \operatorname{div} \mathbf{q}_s - \rho_s Z R_s] \\ &= \rho_f Z g \mathbf{j} \cdot (\mathbf{v}_f - \mathbf{v}_s) + D(Z)(\mathbf{v}_f - \mathbf{v}_s)^2 - \tfrac{1}{2} C \rho_f Z\, d(\mathbf{v}_f - \mathbf{v}_s)^2/dt. \end{aligned} \tag{18}$$

The formal limit $Z \to 0$, or $Z \to 1$, gives the usual energy equation when these limits are appropriately interpreted: the limit $\mathbf{v}_f \to \mathbf{v}_s$ is the correct limit in each case and the right of equation (18) tends to zero.

The individual energy equations require a temperature term which is a function of $(T_s - T_f)$, comparable to the drag term in (12). We shall approximate this term by $(k/\mu)\,D(Z)\,(T_s - T_f)$, where $D(Z)$ is given by (12) and k is the fluid conductivity. As in § 2 we shall put p_s equal to zero. In this case, the consistent fluid pressure term in (18) is $(1-Z)^{-1}\,dp(1-Z)/dt_f$. The approximate energy equations for the solids and fluid respectively are thus given by

$$\rho_s Z\,d(c_s T_s)/dt_s - Z\Phi_s - Z\,\mathrm{div}\,\mathbf{q}_s - \rho_s Z R_s + (k/\mu)\,D(Z)\,(T_s - T_f)$$
$$+\rho_f Z g\mathbf{j}\,.\,\mathbf{v}_s + D(Z)\,\mathbf{v}_s\,.\,(\mathbf{v}_f - \mathbf{v}_s) + C\rho_f Z\mathbf{v}_s\,.\,d(\mathbf{v}_f - \mathbf{v}_s)/dt = 0, \tag{19}$$

$$\rho_f(1-Z)\,d(c_p T_f)/dt_f - (1-Z)^{-1}\,dp(1-Z)/dt_f - (1-Z)\,\Phi_f - (1-Z)\,\mathrm{div}\,\mathbf{q}_f$$
$$-\rho_f(1-Z)\,R_f - (k/\mu)\,D(Z)\,(T_s - T_f) - \rho_f Z g\mathbf{j}\,.\,\mathbf{v}_f - D(Z)\,\mathbf{v}_f\,.\,(\mathbf{v}_f - \mathbf{v}_s)$$
$$-C\rho_f Z\mathbf{v}_f\,.\,d(\mathbf{v}_f - \mathbf{v}_s)/dt = 0. \tag{20}$$

The equation of state for the fluid is taken as

$$p = \rho_f \mathscr{R} T_f, \tag{21}$$

where $\mathscr{R} = c_p - c_v$, c_v being the specific heat, at constant volume, of the fluid. (If p_s were not zero a comparable equation would obtain for it.)

Again, these equations are not restricted to fluidized beds, but apply to general dispersed two-phase flow.

The conservation equations (1), (2), the momentum equations (13), (14), the energy equations (19), (20) and the equation of state (21) form a closed set of equations for $\mathbf{v}_s$, $\mathbf{v}_f$, p, Z, T_s, T_f, ρ_f. When variations in T_s, T_f, ρ_f and ρ_s ($\rho_s \gg \rho_f$) are negligible the energy and state equations are not necessary and the appropriate equations for gas fluidized systems are approximated by (15). The remainder of this paper will be concerned with finding solutions to the equations derived above.

4. Propagation of small disturbances in incompressible beds

We shall refer to incompressible beds as those in which compressibility and temperature effects are negligible, and so ρ_f is taken to be constant. We also take ρ_s to be constant and restrict ourselves to the case where $\mathbf{F}_f \equiv 0 \equiv \mathbf{F}_s$. We shall consider, in this section, the stability of such beds when subjected to a small internal disturbance. The general viscous equations (those equations which include the stress tensors) are studied for general $R = \rho_s/\rho_f$ (> 1). Generally $R \gg 1$ for gas fluidized beds and a good approximation is given by the limiting case $R \to \infty$.

We consider the propagation of small two-dimensional† disturbances superposed only on the steady state $\mathbf{v}_s = 0$. Cartesian co-ordinates x, y are used with y being measured positively in the vertical direction. Unit vectors in the x-, y-direction are $\mathbf{i}$, $\mathbf{j}$ respectively. The bottom of the bed is at $y = 0$ and the surface is at $y = y_0$. From (1), (2), (9) and (13) with ρ_f, ρ_s constant and $\mathbf{F}_f \equiv 0 \equiv \mathbf{F}_s$

† It is most unlikely that three-dimensional disturbances would give any fundamentally different motions.

a steady-flow state with $\mathbf{v}_s = 0$ and all other quantities constant except the pressure, is

$$\left.\begin{aligned} \mathbf{v}_s &= 0, \quad \mathbf{v}_f = (0, v_{f_2}) = (0, v_0), \\ Z &= Z_0, \\ p &= (p)_0 = p_0 - g\rho_s Z_0 y, \\ &\qquad = p_0 - D(Z_0) v_0 y, \end{aligned}\right\} \tag{22}$$

where v_0, Z_0, $(p)_0$, and p_0 are defined by these equations. What is effectively another steady state with Z_0 not constant is studied in § 7, the centrifuge case. Experimentally Z_0 appears to be constant in the fluidized beds discussed in this section (excepting bubbles).

We now consider small perturbations about the steady state given by equation (22) and, since only constant coefficients arise, we anticipate the exponential character of the solutions by writing

$$\left.\begin{aligned} \mathbf{v}_s &= v_0 \mathbf{v}'_s E, \quad \mathbf{v}_f = v_0[\mathbf{j} + \mathbf{v}'_f E], \\ Z &= Z_0[1 + Z'E], \quad p = (p)_0 + p_0 p'E, \\ E &= \exp[i\delta(x - ct) + \lambda(y - y_0)], \end{aligned}\right\} \tag{23}$$

where the accent denotes dimensionless perturbation quantities, δ and λ are wave-numbers, c is a velocity and t is the time. Substitution of relations (23) into (1), (2), (9) and (13) with ρ_f, ρ_s constant, $\mathbf{F}_f \equiv 0 \equiv \mathbf{F}_s$ and using (22) gives the following dimensionless first-order perturbation equations:

$$zv'_{s_2} + iv'_{s_1} - i\omega NZ' = 0, \tag{24}$$

$$zv'_{f_2} + iv'_{f_1} - D(z - i\omega N) Z' = 0, \tag{25}$$

$$\begin{aligned} & v'_{f_2}\left[\left\{\frac{z}{RDN} - \frac{i\omega}{RD} - \frac{\theta_2}{D}(z^2-1) - \frac{\theta_1}{D}z^2\right\}\mathbf{j} - \left\{iz\frac{\theta_1}{D}\right\}\mathbf{i}\right] \\ & + v'_{f_1}\left[-\left\{iz\frac{\theta_1}{D}\right\}\mathbf{j} + \left\{\frac{z}{RDN} - \frac{i\omega}{RD} - \frac{\theta_2}{D}(z^2-1) + \frac{\theta_1}{D}\right\}\mathbf{i}\right] \\ & - v'_{s_2}[\{i\omega + \Pi_2(z^2-1) + \Pi_1 z^2\}\mathbf{j} + \{iz\Pi_1\}\mathbf{i}] \\ & - v'_{s_1}[\{iz\Pi_1\}\mathbf{j} + \{i\omega + \Pi_2(z^2-1) - \Pi_1\}\mathbf{i}] + \frac{R-1}{R}NZ'\mathbf{j} + [z\mathbf{j} + i\mathbf{i}]\frac{p'P}{RD} = 0, \end{aligned} \tag{26}$$

$$\begin{aligned} & -\mathbf{v}'_f\left[N + \frac{C}{N(R-1)}(z - i\omega N)\right] + v'_{s_2}\left[\left\{N - \frac{i\omega R}{R-1} - \Pi_2(z^2-1)\frac{R}{R-1}\right.\right. \\ & \qquad \left.\left. + \frac{C}{N(R-1)}(z - i\omega N) - \Pi_1 z^2 \frac{R}{R-1}\right\}\mathbf{j} - \left\{iz\Pi_1\frac{R}{R-1}\right\}\mathbf{i}\right] \\ & + v'_{s_1}\left[-\left\{iz\Pi_1\frac{R}{R-1}\right\}\mathbf{j} + \left\{N - \frac{i\omega R}{R-1} - \Pi_2(z^2-1)\frac{R}{R-1}\right.\right. \\ & \qquad \left.\left. + \frac{C}{N(R-1)}(z - i\omega N) + \Pi_1\frac{R}{R-1}\right\}\mathbf{i}\right] - N\mathscr{D}Z'\mathbf{j} = 0, \end{aligned} \tag{27}$$

where, we have used the fact from (13) in the steady state that

$$g(\rho_s - \rho_f) = (\mu a H/m)\rho_s(1 + GD_{s_0}) v_0 = D(Z_0) v_0/Z_0, \tag{28}$$

and the dimensionless parameters

$$\left.\begin{aligned} &z = \lambda/\delta, \quad \omega = \delta c/(\delta g)^{\frac{1}{2}}, \quad D = Z_0/(1-Z_0), \quad D_{s_0} = Z_0/(Z_s - Z_0), \\ &\mathscr{D} = GD_{s_0}(1+D_{s_0})/(1+GD_{s_0}), \quad \theta_2 = (\mu^2\delta^3/g\rho_s^2)^{\frac{1}{2}}, \quad \theta_1 = (\tfrac{4}{3}+\zeta/\mu)\,\theta_2, \\ &\Pi_2 = AD_{s_0}\theta_2, \quad \Pi_1 = D_{s_0}(\tfrac{1}{3}A + BD_{s_0}^2)\,\theta_2, \quad N = (g/\delta v_0^2)^{\frac{1}{2}}, \\ &P = (\delta/g)^{\frac{1}{2}} p_0/V, \quad V = \rho_f(1-Z_0)\,v_0, \quad R = \rho_s/\rho_f. \end{aligned}\right\} \tag{29}$$

The dispersion relationship for equations (24)–(27) reduces to

$$(z^2-1)\,L_1(\omega,z)\,L_2(\omega,z) = 0, \tag{30}$$

where

$$\begin{aligned} L_1(\omega,z) = \omega^2\left[1+\frac{C(1+D)}{R}\right] + i\omega\left[N(D+1)\frac{R-1}{R} + (\Pi_1+\Pi_2)(1-z^2)\right. \\ \left. + \frac{C(1+2D)z}{RN}\right] - z\frac{R-1}{R}\left[D+\mathscr{D}+\frac{CDz}{N^2(R-1)}\right], \end{aligned} \tag{31}$$

$$\begin{aligned} L_2(\omega,z) = \omega^2\left[\frac{C(RD+1)+R}{(R-1)RD}\right] + \frac{i\omega}{(R-1)RD}\left[\frac{z}{N}(R+2C+RCD)\right. \\ \left. + N(R-1)(1+RD) - R(z^2-1)\{\theta_2(C+R)+\Pi_2(1+CD)\}\right] \\ + \frac{1}{(R-1)RD}\left[-z(R-1) - \frac{Cz^2}{N^2} + \left\{\frac{Cz}{N} + (R-1)N\right\}\{\theta_2 + D\Pi_2\}R(z^2-1)\right. \\ \left. + \frac{R\Pi_2 z}{N}(z^2-1) - \theta_2\Pi_2(z^2-1)^2R^2\right]. \end{aligned} \tag{32}$$

The (ω, z) relations which correspond to the physically possible motions are given by

$$L_1(\omega,z) = 0, \quad L_2(\omega,z) = 0. \tag{33}$$

If z (effectively the dimensionless y-component wave-number) is given and is imaginary, equations (33), each quadratic in ω, give the ω's and hence the wave speeds and growth rates of small disturbances propagated in the interior of the bed.

For general R not near the limiting values unity and infinity there appears one ω from each of equations (33) which represents a wave which initially grows exponentially. The remaining two ω's represent attenuating waves. Let ω_{01}, ω_{11} be the solutions of the first of equations (33), and ω_{02}, ω_{12} the solutions of the second of equations (33), the zero subscript ω's representing the growing waves. At either end of the R scale there is only one unstable mode. As $R \to \infty$, ω_{01} persists as the unstable mode and ω_{02} disappears, whilst as $R \to 1+$, ω_{02} persists and ω_{01} disappears. In the latter case the expression (31) becomes linear in ω, and in the former, expression (32) becomes linear in ω; each linear form equated to zero gives only the stable oscillation ω_{11} or ω_{12}.

The phenomena to be discussed in a later paper are related to beds where $R \gg 1$. A good approximation is given by $R \to \infty$ and it is therefore appropriate to discuss this case in detail, in which event

$$\left.\begin{aligned} \omega_{01}, \omega_{11} = \tfrac{1}{2}i[\{[N(D+1)+(\Pi_1+\Pi_2)(1-z^2)]^2 - 4z(D+\mathscr{D})\}^{\frac{1}{2}} \\ -\{N(D+1)+(\Pi_1+\Pi_2)(1-z^2)\}], \\ \omega_{12} = -i\Pi_2(1-z^2), \end{aligned}\right\} \tag{34}$$

where the appropriate branch is taken in the square root so that ω_{01} has a positive imaginary part. It is the above ω_{01}-wave or that from the first of equations (33) with $R \gg 1$ which probably gives the linearized description of the way in which a bubble (or void) starts and rises in fluidized beds in which $R \gg 1$. Solving for Z' and p' (see § 5) we find that when $Z' > 0, p' < 0$ and so a solids condensation is accompanied by a decrease in pressure, and vice versa.

Equations (31)–(33) furnish a way in which general fluidized beds may be classified. For given bed characteristics the mode with the largest growth rate would be the reasonable one to study and provide consistent numbers for a given z. Thus no single solution, neither ω_{01} nor ω_{02}, would give a reasonable classification over the whole R-range of interest.

As mentioned in the introduction there is a fundamental experimental and observational difference in beds when $R > 10$ and $R < 10$ approximately. The former type of bed has bubbles of fluid rising up through the bed whilst the latter does not. From observation in the $R < 10$ case the particles and fluid are turbulent and any bubble of fluid injected into the bed is rapidly dispersed. The above linear analysis gives no indication as to whether a bubble will or will not form, it states only that small internal disturbances will initially grow exponentially. In fluidized beds with $R \gg 1$ there is only one unstable mode of interest whilst for $R < 10$ there are two possible unstable modes of interest. Although this may be some indication that the two régimes are basically different it is clear that bubble formation usual in most gas fluidized beds and the 'non-bubble' or turbulent motion prevalent in most liquid fluidized beds is essentially a non-linear effect.

5. Surface wave propagation in incompressible beds

Experimentally, when bubbles exist in a bed, they generate surface waves on breaking the surface. These surface waves attenuate very rapidly and have practically disappeared within a few wavelengths. Observation of surface waves in very shallow beds may be difficult and the results confusing and inconclusive because bubbles may span the depth of the bed and result in what appears to be channels through which only fluid passes. However, when 'genuine' surface waves are generated they are also stable. In the case when beds cannot sustain bubbles, surface waves are also quickly damped. Thus, except possibly in very shallow beds, small surface waves appear to be highly stable.

Equations (30)–(33) are a set of relations which give various z's as functions of ω. There are eight roots for z. Further relations, which come from appropriate surface conditions are required so that the ω's and z's may be determined. Clearly the only physically realistic ω are those which give $\mathrm{Re}\, z > 0$, otherwise the conditions at the bottom of the bed could not be satisfied. Clearly all z's in equations (30)–(33) are not allowable. For example, $z = -1$ is excluded.

Surface conditions are obtained as follows. Let $y = y_0 + \zeta(x,t)$ be the free surface. If we integrate the momentum equation (9), with ρ_f, ρ_s constant and $\mathbf{F}_f \equiv 0 \equiv \mathbf{F}_s$, between $y = y_0$ and $y = y_0 + \zeta$ we get, to first order,

$$p(y_0+\zeta) - p(y_0) + [\rho_f(1-Z_0) + \rho_s Z_0] g\zeta = 0. \tag{35}$$

$p(y_0+\zeta)$ is the ambient pressure above the bed. Consistent with (23),

$$p(y_0) = [(p)_0]_{y=y_0} + p_0 p' \exp i\delta(x-ct),$$

where the first term on the right is the ambient pressure above the bed. If we write

$$\zeta = \zeta' \exp i\delta(x-ct),$$

and use the last two equations and (35) we get

$$p_0 p' - [\rho_f(1-Z_0)+\rho_s Z_0] g\zeta' = 0.$$

At a free surface we also have, to first order

$$\partial\zeta/\partial t = v_{s_2} \Rightarrow v_0 v'_{s_2} + i\delta c\zeta' = 0,$$

which, combined with the $p'-\zeta'$ equation above gives the kinematic condition

$$v'_{s_2} + i\omega(p'P/RD)[1+1/RD]^{-1} = 0. \tag{36}$$

Further surface conditions on the particulate phase are those which require the normal and tangential stress tensor to be zero. Thus, from (8),

$$\sigma_{s_{ii}} = 0 \Rightarrow \operatorname{div} \mathbf{v}_s = 0 \qquad \Rightarrow zv'_{s_2} + iv'_{s_1} = 0, \tag{37}$$

$$\sigma_{s_{12}} = 0 \Rightarrow \partial v_{s_1}/\partial y + \partial v_{s_2}/\partial x = 0 \quad \Rightarrow iv'_{s_2} + zv'_{s_1} = 0, \tag{38}$$

on the surface. Note that the boundary condition (37) would be different if we had neglected $\frac{2}{3}A$ compared with BD_s^2 in (8). As shown below the form of (37) is essential to the solution. The $v'_{s_2}, v'_{s_1}, p'P/RD$ as functions of z, ω are obtained from (24)–(27). Since the determinant of the coefficients is zero (equation (30)) the $v'_{s_2}, v'_{s_1}, p'P/RD$ are proportional to any corresponding set of cofactors of these terms in the determinant. Any set (from any one row) which does not give $v'_{s_2}, v'_{s_1}, p'P/RD$ all identically zero for any specific z from equation (30) is allowable. Denote the solutions for each z by $f_{v_{s2}}[\omega, z(\omega)], f_{v_{s1}}[\omega, z(\omega)], f_p[\omega, z(\omega)]$ respectively. These are, in general, functions of ω and $z(\omega)$: the latter are from (30)–(33). If there were only one z, then (36)–(38) would give the possible ω if the equations were consistent. However, since there are several z from (30), (33) the solutions for $v'_{s_2}, v'_{s_1}, p'P/RD$ are given by

$$\left.\begin{aligned} v'_{s_2} &= \sum_{j=1} A_j f_{v_{s2}}[\omega, z_j(\omega)], \\ v'_{s_1} &= \sum_{j=1} A_j f_{v_{s1}}[\omega, z_j(\omega)], \\ \frac{p'P}{RD} &= \sum_{j=1} A_j f_p[\omega, z_j(\omega)], \end{aligned}\right\} \tag{39}$$

where the A_j are constants which have to be determined. The summation for j is over the number of z's with positive real part from (30)–(33) determined *a posteriori*. Thus, with (39), the appropriate boundary conditions (36)–(38) give

$$\left.\begin{aligned} \sum_j A_j\{f_{v_{s2}}[\omega, z_j(\omega)] + i\omega(1+1/RD)^{-1} f_p[\omega, z_j(\omega)]\} &= 0, \\ \sum_j A_j\{z_j f_{v_{s2}}[\omega, z_j(\omega)] + if_{v_{s1}}[\omega, z_j(\omega)]\} &= 0, \\ \sum_j A_j\{if_{v_{s2}}[\omega, z_j(\omega)] + z_j f_{v_{s1}}[\omega, z_j(\omega)]\} &= 0. \end{aligned}\right\} \tag{40}$$

Equations (40) give a set of equations for the A_j. The appropriate conditions on the coefficients for a non-trivial solution provide the relationships between the allowable ω in terms of the z_j. Equations (33) with these $\omega(z_j)$ substituted then provide the corresponding z_j. It is at this stage that the number of allowable z_j (that is with $\operatorname{Re} z_j > 0$) is found.

Using the cofactors of the first row in the determinant of the coefficients in equations (24)–(27), and introducing

$$\left.\begin{aligned} U &= RD/(1+RD), \quad W = N+\{C/N(R-1)\}(z-i\omega N), \\ X &= \{R/(R-1)\}[i\omega+\Pi_2(z^2-1)]-\{C/N(R-1)\}(z-i\omega N)-N, \\ Y &= (1/RDN)(z-i\omega N)-\theta_2(z^2-1)/D, \end{aligned}\right\} \tag{41}$$

we get, apart from a constant independent of ω and z,

$$\begin{aligned} f_{v_{s2}}+Ui\omega f_p = {} & (1-z^2)L_2(\omega,z)[Ui\omega^2R\theta_1\Pi_1(1-z^2)/D(R-1) \\ & -U\theta_1(i\omega^2X+\omega z\mathscr{D})/D+Ui\Pi_1\omega^2\{W+RY/(R-1)\}] \\ & -[Ui\omega^2\{L_2(\omega,z)\}^2+U\omega zL_2(\omega,z)\{W/R+W(1-U)/U+Y\mathscr{D}\} \\ & +iW\{(R-1)W/R-Y\mathscr{D}\}], \end{aligned} \tag{42}$$

$$zf_{v_{s2}}+if_{v_{s1}} = W(z^2-1)L_2(\omega,z), \tag{43}$$

$$if_{v_{s2}}+zf_{v_{s1}} = W[2i\omega zL_2(\omega,z)-(1+z^2)\{(R-1)W/R-Y\mathscr{D}\}]. \tag{44}$$

From the first of equations (33) using (31) there is only one $z = z_1$, say, which has $\operatorname{Re} z_1 > 0$. Also $L_2(\omega, z_1) \neq 0$. Further $z = +1$ is an allowable solution of (30). Because of the form of (43) all z_j except $z = z_1$ make this term zero and the consistency of equation (40) thus reduces to requiring all two by two determinants of the matrix

$$\begin{bmatrix} f_{v_{s2}}(\omega,1)+Ui\omega f_p(\omega,1), & \ldots, & f_{v_{s2}}(\omega,z_j)+Ui\omega f_p(\omega,z_j), & \ldots \\ if_{v_{s2}}(\omega,1)+f_{v_{s1}}(\omega,1), & \ldots, & if_{v_{s2}}(\omega,z_j)+z_jf_{v_{s1}}(\omega,z_j), & \ldots \end{bmatrix}$$

to be zero, where j goes from $j = 2$, and the z_j $(j \neq 1)$ are solutions of the second of equations (33), $L_2(\omega, z_j) = 0$. *Note* that if we had omitted the shear viscosity contribution in $\sigma_{s_{ij}}$ compared with the bulk viscosity contribution this simplification would not have been possible. In fact, in the particular case $(R \to \infty)$ evaluated below, it would have predicted an unstable surface oscillation! Thus The forms of (42), (44) which are used in the above matrix become, on extracting appropriate non-zero factors,

$$\left.\begin{aligned} [f_{v_{s2}}+Ui\omega f_p]_{z=z_j} &= -i\{W[(R-1)W/R-Y\mathscr{D}]\}_{z=z_j}, \\ [if_{v_{s2}}+zf_{v_{s1}}]_{z=z_j} &= -\{(1+z^2)W[(R-1)W/R-Y\mathscr{D}]\}_{z=z_j}. \end{aligned}\right\} \tag{45}$$

When $z = 1$ the last square bracket in (42) has as a factor half the square bracket in (44), that is

$$\begin{aligned} & [Ui\omega^2\{L_2(\omega,1)\}^2+U\omega L_2(\omega,1)\{W/R+W(1-U)/U+Y\mathscr{D}\}_{z=1} \\ & \qquad +i\{(R-1)W/R-Y\mathscr{D}\}_{z=1}] \\ & = [i\omega L_2(\omega,1)-\{(R-1)W/R-Y\mathscr{D}\}_{z=1}][U\omega L_2(\omega,1)-i\{W\}_{z=1}]. \end{aligned}$$

The consistency condition from the matrix now reduces to requiring all two by two determinants, always using the first column, of the following matrix to be zero:

$$\begin{bmatrix} -U\omega L_2(\omega,1)+i\{W\}_{z=1}, & i, & i, & \dots, & i \\ 2\{W\}_{z=1}, & (1+z_2^2), & (1+z_3^2), & \dots, & (1+z_j^2) \end{bmatrix}.$$

Thus

$$U\omega L_2(\omega,1)(1+z_j^2)+i\{W\}_{z=1}(1-z_j^2) = 0, \tag{46}$$

for all z_j ($z \neq 1$) with $\operatorname{Re} z_j > 0$, where the z_j are solutions of $L_2[\omega(z),z] = 0$. Equations (46) gives $\omega = \omega(z_j)$ which when substituted into (33) give the z_j, including the $j = 1$ case from $L_1(\omega,z) = 0$.

For given bed constants (46) with (30)–(33) give the surface modes of oscillation and the corresponding wave-numbers. For general R the algebraic equations are complicated but numerical solutions would be easily found.

We shall consider in particular the case of fluidized beds when $R \to \infty$ which can be solved algebraically exactly. The equations (30)–(33) imply

$$\left.\begin{aligned} z &= \pm 1, \\ \omega^2+i\omega[N(D+1)+(\Pi_1+\Pi_2)(1-z^2)]-z(D+\mathscr{D}) &= 0, \\ i\omega[1-(\theta_2/ND)(z^2-1)]+[\Pi_2(z^2-1)+(\theta_2/ND)(z^2-1)\{N-\Pi_2(z^2-1)\}] &= 0. \end{aligned}\right\} \tag{47}$$

Equation (46) reduces to

$$z_j^2-1 = 2\omega^2/(1-\omega^2), \tag{48}$$

where the z_j are solutions of the last of equations (47) with $\operatorname{Re} z_j > 0$. If we substitute (48) in the last of equations (47) we find that the algebraic equation in $\eta = -i\omega$ is a polynomial with real positive coefficients, namely,

$$\eta\left[1+\frac{\theta_2\, 2\eta^2}{ND(1+\eta^2)}\right]+\left[\frac{\Pi_2\, 2\eta^2}{1+\eta^2}+\frac{\theta_2\, 2\eta^2}{ND(1+\eta^2)}\left\{N+\frac{\Pi_2\, 2\eta^2}{1+\eta^2}\right\}\right] = 0, \tag{49}$$

which implies that $\operatorname{Re}\eta < 0$. Thus all surface modes of oscillation are stable since $\operatorname{Im}\omega < 0$. Equation (49) determines η which on substitution in (48) gives the z_j.

Since $\Pi_2 \gg \theta_2$, in general in beds where $R \gg 1$, we find in this case that the surface modes are given by

$$\left.\begin{aligned} &\omega = -i\Pi_2 \pm i[\Pi_2^2-1]^{\frac{1}{2}}, \quad z = 1, \\ &z_1 = [-i/2\omega(\Pi_1+\Pi_2)] \\ &\quad \times [\{(D+\mathscr{D})^2+4i\omega(\Pi_1+\Pi_2)[\omega^2+i\omega\{N(D+1)+\Pi_1+\Pi_2\}]\}^{\frac{1}{2}}-(D+\mathscr{D})], \\ &j = 2, \quad z_2 = (1-1/\Pi_2^2)^{\frac{1}{2}} \quad \text{if} \quad \Pi_2 > 1 \\ &\qquad\quad\;\; = (1/\Pi_2^2-1)^{\frac{1}{2}} i^{\frac{1}{2}} \quad \text{if} \quad \Pi_2 < 1. \end{aligned}\right\} \tag{50}$$

Note that, since $\Pi_2 \approx O(\delta^{\frac{3}{2}})$ and $\omega \approx O(\delta^{\frac{1}{2}})$, the attenuating factor of such a wave is of $O(\delta^2 t)$.

Although it would require checking in specific cases, it is unlikely that the inclusion of finite R (not in the region of unity) would introduce any new mechanism which would result in unstable surface oscillations. As shown in § 4 no fundamentally new phenomena resulted from the inclusion of finite R.

6. Internal stability of hot beds with $R \gg 1$

In this section we shall consider the propagation of small two-dimensional disturbances (as in § 4) in a gas fluidized bed with $R \gg 1$ where compressibility and heat effects are large. This could arise from reaction heat in the bed or from externally applied heat. We consider the case ρ_s constant and $\mathbf{F}_f \equiv 0 \equiv \mathbf{F}_s$. The conservation equations (1) and (2) are used, with ρ_s omitted in (1), and the approximate momentum equations are the third and fourth of (15).

A practical case of importance in an application to a nuclear power source is that in which R_s is large compared with any other heat source. In fact gas temperature changes of $O(10^3\,\mathrm{K})$ are anticipated. The case we consider here is that in which Φ_f, $\operatorname{div}\mathbf{q}_f$, R_f, Φ_s, $\operatorname{div}\mathbf{q}_s$, $\rho_f Zg\mathbf{j}.(\mathbf{v}_f-\mathbf{v}_s)$, $D(Z)(\mathbf{v}_f-\mathbf{v}_s)^2$ and $C\rho_f Z\,d(\mathbf{v}_f-\mathbf{v}_s)^2/dt$ are small compared with R_s and will be neglected in the energy equations: we shall use the two equations (18) and (20). For convenience of reference the equations we use are the following (from (1), (2), (15), (18), (20), (21)):

$$\left.\begin{aligned}
\operatorname{div} Z\mathbf{v}_s &= -\partial Z/\partial t,\\
\operatorname{div}\rho_f(1-Z)\mathbf{v}_f &= -\partial\rho_f(1-Z)/\partial t,\\
\rho_s Z\,d\mathbf{v}_s/dt_s &= -g\rho_s Z\mathbf{j} + Z\operatorname{div}\boldsymbol{\sigma}_s + D(Z)(\mathbf{v}_f-\mathbf{v}_s),\\
\operatorname{grad} p &= -D(Z)(\mathbf{v}_f-\mathbf{v}_s),\\
\rho_f(1-Z)\,d(c_pT_f)/dt_f + \rho_s Z\,d(c_sT_s)/dt_s - (1-Z)^{-1}d(1-Z)p/dt_f - \rho_s ZR_s &= 0,\\
\rho_s Z\,d(c_sT_s)/dt_s - \rho_s ZR_s + (k/\mu)D(Z)(T_s-T_f) &= 0,\\
p &= \rho_f\mathcal{R}T_f.
\end{aligned}\right\} \quad (51)$$

We shall treat (as in § 4) only superpositions on the steady state $\mathbf{v}_s \equiv 0$, $\partial/\partial t = \partial/\partial x = 0$, $\mathbf{v}_f = (0, v)$. All steady-state quantities are functions of y only. Equations (51) in the steady state give

$$\left.\begin{aligned}
\rho_f(1-Z)v = V &= \text{const.},\\
g\rho_s Z &= D(Z)v,\\
dp/dy &= -D(Z)v,\\
\rho_f(1-Z)v\,d(c_pT_f)/dy - [v/(1-Z)]\,d(1-Z)p/dy - \rho_s ZR_s &= 0,\\
\rho_s ZR_s &= (k/\mu)D(Z)(T_s-T_f),\\
p &= \rho_f\mathcal{R}T_f.
\end{aligned}\right\} \quad (52)$$

If we anticipate that the pressure drop through the bed is to be of the order of one atmosphere then

$$\frac{v}{1-Z}\frac{d(1-Z)p}{dy}\bigg/\rho_f(1-Z)v\frac{d(c_pT_f)}{dy} \approx O(10^{-2}).$$

(Higher pressure drops imply deeper beds with larger temperature variation.) We shall therefore neglect the pressure term in the first of the energy equations in (52) and it becomes

$$Vc_p\,dT_f/dy = \rho_s ZR_s.$$

We shall adopt the approximation that μ and k depend linearly on the temperature. Thus

$$\mu/\mu(0) = T_f/T_f(0) = k/k(0). \tag{53}$$

We introduce

$$\left.\begin{aligned} M &= \int_0^y \rho_s Z\,dy, & \Gamma &= M(y_0) = \int_0^{y_0} \rho_s Z\,dy, \\ \xi &= M/\Gamma, & \tau &= y\rho_s/\Gamma, \quad \gamma = \Gamma g/p(0), \\ \alpha &= R_s\Gamma/c_p V T_f(0), & \beta &= T_f(0)\mathscr{R}\mu(0)\,aH/mgp(0). \end{aligned}\right\} \tag{54}$$

Γg is the weight per unit area of the bed. From (54)

$$Z = d\xi/d\tau. \tag{55}$$

Since we are interested in the gross features of the phenomenon we further approximate by choosing in (12)

$$G = 1, \quad Z_s = 1 \Rightarrow D(Z) = \mu a H \rho_s Z/m(1-Z). \tag{56}$$

Thus, (50)–(55) give the following approximate steady-state equations for hot beds:

$$\left.\begin{aligned} \rho_f(1-Z)v = V, \quad \mathbf{v}_s = 0, \quad \mathbf{v}_f = (0,v), \\ mg = \mu(0)\,T_f aHv/T_f(0)\,(1-Z), \quad p + Mg = p(0), \\ c_p V[T_f - T_f(0)] = MR_s, \quad T_s = mR_sT_f(1-Z)/k(0)\,T_f(0)\,aH + T_f, \quad p = \rho_f\mathscr{R}T_f, \end{aligned}\right\}$$

which have dimensionless solutions

$$\left.\begin{aligned} p/p(0) = 1-\gamma\xi, \quad T_f/T_f(0) = 1+\alpha\xi, \\ \rho_f/\rho_f(0) = (1-\gamma\xi)/(1+\alpha\xi), \quad v/v(0) = 1/(1-\gamma\xi)^{\frac{1}{2}}, \\ Z = d\xi/d\tau = 1 - \beta^{\frac{1}{2}}(1+\alpha\xi)/(1-\gamma\xi)^{\frac{1}{2}}. \end{aligned}\right\} \tag{57}$$

Since $\alpha \gg 1$, $\alpha \gg \gamma$ the last of equations (57) gives an approximate solution

$$\xi = [(1-\beta^{\frac{1}{2}})/\alpha\beta^{\frac{1}{2}}][1 - \exp(-\alpha\beta^{\frac{1}{2}}\tau)], \quad Z = (1-\beta^{\frac{1}{2}})\exp(-\alpha\beta^{\frac{1}{2}}\tau), \tag{58}$$

which shows that an increase in the heat input (R_s or α) produces an increased variation (a decrease with height) in Z. This is useful in the nuclear power source application because it implies that the hottest particles at the top of the bed are farthest apart. Note that the general case when $D(Z)$ is that given by (12), rather than (56), does not change the qualitative result. A comparable result to (58) was first found by Carrier & Cashwell (1956) but there $D(Z)$ is taken to be constant as compared with (56) above: a more complicated form than (58) results!

We now consider the superposition of a small two-dimensional disturbance on the steady state (57). We denote steady-state quantities by subscript zero. Because the steady-state quantities are functions of y (or ξ) the usual linearization would not result in convenient constant multipliers of an exponential factor as above. Since we are seeking only the gross features we shall approximate the variable multipliers of the perturbed quantities by suitable average constant values and accordingly use exponential factors as in § 4. Clearly, first estimates will be obtained and questions as to the stability of the bed answered. As before, then, we write†

$$\left.\begin{aligned} \mathbf{v}_{f_0} &= v_0\mathbf{j}, & \mathbf{v}_f &= v_0(\mathbf{j}+\mathbf{v}'_f E),\\ \rho_f &= \rho_{f_0}(1+\rho'_f E), & T_f &= T_{f_0}(1+T'_f E),\\ Z &= Z_0(1+Z'E), & T_s &= T_{s_0}(1+T'_s E),\\ \mathbf{v}_s &= v_0\mathbf{v}'_s E, & \mu &= \mu_0(1+T'_f E),\\ p &= p_0(1+p'E), & k &= k_0(1+T'_f E), \end{aligned}\right\} \tag{59}$$

with E from (23). Substitution of (59) into (51) gives rise to terms like dv_0/dy, dp_0/dy, etc. We shall retain these expressions until the end of any algebraic manipulation and then replace them by average values if the motion is internal, and more appropriately by their value at $\xi = 1$ (the surface) for surface waves. Average values are obtained simply from (57) by integrating them as functions of ξ from 0 to 1. Thus, in what follows, it is to be remembered that all derivatives with respect to y (or ξ) of steady-state functions are to be read as average values.

The dimensionless parameters in equations (29) will appear except that $D_{s_0} = D = \mathscr{D}$ in this case and all parameters are averaged, where necessary, throughout the bed as described above. We introduce further parameters which pertain to the variations in the steady-state variables, namely

$$\left.\begin{aligned} &K_1 = \frac{1}{\delta p_0}\frac{dp_0}{dy} = \frac{Z_0\rho_s}{\delta\Gamma}\left(\frac{-\gamma}{1-\gamma\xi}\right), \quad K_2 = \frac{1}{\delta T_{f_0}}\frac{dT_{f_0}}{dy} = \frac{Z_0\rho_s}{\delta\Gamma}\left(\frac{\alpha}{1+\alpha\xi}\right),\\ &K_3 = \frac{1}{\delta v_0}\frac{dv_0}{dy} = \frac{Z_0\rho_s}{\delta\Gamma}\left[\frac{\gamma}{2(1-\gamma\xi)}\right], \quad K_4 = \frac{1}{\delta v_0 Z_0}\frac{d(v_0Z_0)}{dy},\\ &K_5 = \frac{1}{\delta}\frac{dD}{dy} = -\frac{\rho_s Z_0(1-\gamma\xi)^{\frac{1}{2}}[2\alpha(1-\gamma\xi)+\gamma]}{\delta\Gamma\beta^{\frac{1}{2}}\,2(1+\alpha\xi)^2},\\ &K_6 = \frac{1}{\delta\mu_0}\frac{d(\mu_0 D[BD^2+\frac{4}{3}A])}{dy}, \quad K_7 = \frac{1}{\delta^2 v_0}\frac{d^2v_0}{dy^2},\\ &K_8 = \frac{1}{\delta\mu_0}\frac{d(\mu_0 AD)}{dy}, \quad K_9 = -\frac{d(1-Z_0)p_0}{dy}\Big/\rho_{f_0}(1-Z_0)\,c_p\frac{dT_{f_0}}{dy},\\ &K_{10} = \frac{dT_{s_0}}{dy}\Big/\frac{dT_{f_0}}{dy}, \quad \Omega = \delta T_{f_0}\Big/\frac{dT_{f_0}}{dy},\\ &\phi = T_{s_0}/T_{f_0}, \quad Pr = \mu_0 c_p/k_0, \quad c_s/c_p = s. \end{aligned}\right\} \tag{60}$$

Substitution of (59) into (51) and using relations (29) and (60) we get, after some manipulation, the following first-order perturbation equations, where it is understood that all coefficients are replaced by their appropriate corresponding constants:

$$(z+K_4)v'_{s_2} + iv'_{s_1} - i\omega NZ' = 0, \tag{61}$$

$$zv'_{f_2} + iv'_{f_1} + (z-i\omega N)\rho'_f + [D(i\omega N - z) - K_5]Z' = 0, \tag{62}$$

$$\begin{aligned} &-v'_{s_2}[\{i\omega - \Pi_2 + (\Pi_1+\Pi_2)(z^2+2zK_3+K_7) + \theta_2K_6(z+K_3)\}\mathbf{j}\\ &\qquad\qquad + \{i\Pi_1(z+K_3) + i\theta_2K_8\}\mathbf{i}]\\ &-v'_{s_1}[\{i\Pi_1(z+K_3) + i\theta_2(K_6-2K_8)\}\mathbf{j}\\ &\qquad + \{i\omega - (\Pi_1+\Pi_2) + \Pi_2(z^2+2zK_3+K_7) + \theta_2K_8(z+K_3)\}\mathbf{i}]\\ &\qquad + NZ'\mathbf{j} + [(z+K_1)\mathbf{j} + i\mathbf{i}]p'P/RD = 0, \end{aligned} \tag{63}$$

† Where the subscript zero denotes the steady-state solution.

$$N\mathbf{v}'_f - N\mathbf{v}'_s + \{N/(1-Z_0)\}Z'\mathbf{j} + [(z+K_1)\mathbf{j}+i\mathbf{i}]p'P/RD + NT'_f\mathbf{j} = 0, \quad (64)$$

$$v'_{s_2}K_{10} - i\omega\Omega\phi NT'_s = 0, \quad (65)$$

$$v'_{s_2}[Pr(sK_{10}/N^2\Omega)] - T'_f + T'_s[\phi(1-i\omega sPr/N)] = 0, \quad (66)$$

$$\rho'_f + T'_f = 0, \quad (67)$$

where in (65)–(67), since $\phi - 1 \ll 1$, $P \gg 1$ and $P/RD \gg 1$, all terms of $O(\phi-1)$, $O(1/R)$, $O(1/P)$ and $O(RD/P)$ are omitted. In obtaining the first-order energy equation (65) ρ_f-terms cannot be dropped *a priori* in the fifth equation of (51) since it is necessary for the steady-state solution that they be retained. It was found throughout that it was more illuminating to retain all terms until the final non-dimensional stage and then to neglect the small terms.

For convenience we introduce

$$\left.\begin{aligned} I &= \Pi_2 - (\Pi_1+\Pi_2)(z^2+2zK_3+K_7) - \theta_2K_6(z+K_3), \\ J &= i[\Pi_1(z+K_3)+\theta_2K_8], \\ K &= i[\Pi_1(z+K_3)+\theta_2(K_6-2K_8)], \\ L &= (\Pi_1+\Pi_2) - \Pi_2(z^2+2zK_3+K_7) - \theta_2K_8(z+K_3), \end{aligned}\right\} \quad (68)$$

and the dispersion relation for (61)–(67) becomes

$$\begin{aligned} &K_{10}(2z-i\omega N)[i\omega(z+K_1)(L-i\omega)+(1-\omega K)] \\ &\quad + i\omega\Omega[(I-i\omega)\{i\omega N(D+1)-(D+1/(1-Z_0))z - i\omega(L-i\omega)[z(z+K_1)-1]\} \\ &\quad - \{i\omega N(D+1)-(D+1/(1-Z_0))z\}\{iJ(z+K_1)+iK(z+K_4) \\ &\quad + (z+K_1)(z+K_4)(L-i\omega)\} + (1-\omega K)\{K_4N - iJ[z(z+K_1)-1]\} \\ &\quad + i\omega(z+K_1)(L-i\omega)K_4N - [z(z+K_1)-1](z+K_4)(L-i\omega)] = 0. \end{aligned} \quad (69)$$

The square bracket term multiplying $i\omega\Omega$ reduces to

$$\begin{aligned} &\omega^3[i\{z(z+K_1)-1\}] \\ &\quad -\omega^2[N\{z(z+K_1)-1\} + ND\{(z+K_4)(z+K_1)-1\} + (I+L)\{z(z+K_1)-1\}] \\ &\quad -i\omega[Z(D+1/(1-Z_0))\{(z+K_4)(z+K_1)-1\} + N(D+1)\{iJ(z+K_1)+iK(z+K_4) \\ &\quad + (z+K_1)(z+K_4)L - I\} + \{z(z+K_1)-1\}\{IL-JK-(z+K_4)\} \\ &\qquad\qquad - K_4N\{iK+(z+K_1)L\}] \\ &\quad + [z(D+1/(1-Z_0))\{iJ(z+K_1)+iK(z+K_4)+(z+K_1)(z+K_4)L-I\} \\ &\qquad\qquad - \{z(z+K_1)-1\}\{iJ+(z+K_4)L\}+K_4N]. \end{aligned} \quad (70)$$

Equation (69) is a quartic in ω (because of the K_{10}-term $\omega = 0$ is not a solution), which is complicated. However, since our prime interest is the question of stability we shall consider the form where δ is large or small.

In the $\delta \gg 1$ case we may, as a first approximation, consider only the $i\omega\Omega$-term (because $\Omega \approx O(\delta)$) and the dispersion relation is expression (70) equated to zero. With δ large, we may, again to a first approximation, neglect the K_1, K_2, K_3, K_4 (clearly small-wavelength disturbances would not be affected by such bed variations) and from (70) get a crude approximate dispersion relation

$$i\omega\Omega(z^2-1)[i\omega+\Pi_2(z^2-1)][\omega^2+i\omega\{N(D+1)+(\Pi_1+\Pi_2)(1-z^2)\}-2zD] = 0,$$

which gives the same as (34) when $\mathscr{D} = D$: these have an unstable mode. However, a careful order of magnitude argument in δ with $N \approx O(\delta^{-\frac{1}{2}})$, $\Pi_2 \approx O(\delta^{\frac{3}{2}})$, $\Omega \approx O(\delta)$ reduces (69) to

$$\omega^4\Omega(z^2-1) - i\omega^3\Omega(\Pi_1+2\Pi_2)(1-z^2)^2 + \omega^2\Omega\Pi_2(\Pi_1+\Pi_2)(1-z^2)^3 + i\omega\Omega\, 2zD\Pi_2(1-z^2)^2 - 2zK_{10} = 0. \quad (71)$$

If we write $\omega \approx O(\delta^{\epsilon})$ then consistent ϵ from (71) are $\epsilon = \frac{3}{2}, -\frac{3}{2}, -\frac{5}{2}$ with corresponding solutions for ω given from (71) by

$$\left.\begin{aligned} i\omega &= \Pi_2(1-z^2), & i\omega &= (\Pi_1+\Pi_2)(1-z^2), \\ i\omega &= 2zD/(\Pi_1+\Pi_2)(1-z^2), & i\omega &= K_{10}/\Omega D\Pi_2(1-z^2)^2. \end{aligned}\right\} \quad (72)$$

For internal waves z is a given imaginary number, and from equations (57) and (60) the average value to be used for K_{10} is clearly positive as is Ω, and so the first, second and fourth of equations (72) represent stable modes. The third gives a neutrally stable mode. It would appear, then, that the bed is stable. However, if we consider $\delta \gg 1$ in (34) we get exactly the first three of equations (72). The only extra mode is that with K_{10} involved which is stable. Thus, when higher-order terms are included in the derivation of equations (72) it is the neutrally stable third mode which becomes unstable as it does in the no-heat case. In conclusion, therefore, when δ is large there are four possible modes of oscillation, three of which correspond to those in the no-heat case in § 4, one of these three becomes unstable when higher-order δ-terms are included. The fourth mode which only appears in the hot beds is stable and in any event is of $O(\delta^{-\frac{5}{2}})$.

When δ is small the first-order equation for the ω from (69) is (since $K_1, K_3, K_4, K_8 \approx O(\delta^{-1})$)

$$\omega^4[z\Omega K_1] + i\omega^3[NK_1(K_{10}+\Omega DK_4)] + \omega^2[N\theta_2 K_1K_3K_8(K_{10}+\Omega DK_4)] + i\omega[2z\theta_2K_1K_3K_8(K_{10}+\Omega DK_4)] - 2zK_{10} = 0. \quad (73)$$

With $\omega \approx O(\delta^{\epsilon})$, (73) gives consistent values for ϵ of $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ with corresponding solutions for ω from (73) given by

$$\left.\begin{aligned} i\omega &= N(K_{10}+\Omega DK_4)/z\Omega, & i\omega &= -\theta_2K_3K_8, \\ i\omega &= 2z/N, & i\omega &= K_{10}/\theta_2K_1K_3K_8(K_{10}+\Omega DK_4). \end{aligned}\right\} \quad (74)$$

From (57), (60) and the expression for T_s, $K_{10} > 1$, and so

$$K_{10}+\Omega DK_4 > 1+\Omega DK_4 = \gamma(1+\alpha\xi)(2Z_0-1)/2\alpha(1-\gamma\xi)(1-Z_0). \quad (75)$$

If we consider $\alpha \gg \gamma$ the average $K_{10}+\Omega DK_4$ may reasonably be that over ξ from 0 to 1 with $1-\gamma\xi \approx 1$. In this case

$$K_{10}+\Omega DK_4 > 1+\Omega DK_4 \approx (\gamma/2\alpha)[1-\beta^{\frac{1}{2}}(1+\alpha)-\beta^{\frac{1}{2}}].$$

From (57), β (a form of the dimensionless gas flow rate) is such that $1-\beta^{\frac{1}{2}}(1+\alpha) > 0$. Since $Z_0 > 0$ and so for small enough β, which is general in fluidized beds, $K_{10}+\Omega DK_4 > 0$. In most beds $Z_0 > 0{\cdot}5$ (large increases in flow produce bubbles and/or transport) and so from (75), $K_{10}+\Omega DK_4 > 0$ in general. From (57), (60) (neglecting the γ's)

$$K_1 < 0, \quad K_3 > 0, \quad K_8 \approx -A\alpha\rho_s Z_0/\Gamma(1+\alpha\xi) < 0. \quad (76)$$

Their averages are clearly of the same sign.

Thus, from (75) and (76) the modes of oscillation in (74) are all stable except the third, which is neutrally stable. However, in § 4, from (34) ω_{01}, which corresponding to the unstable mode, asymptotes to $-i2z(D+\mathscr{D})/N(D+1)$ when $\delta \ll 1$. In this form it is neutrally stable. Thus, as above in the $\delta \gg 1$ case, the neutrally stable mode in the third of equations (74) will give an unstable mode to higher order in δ which is analogous to that in the incompressible gas fluidized beds (§ 4). Again, it is this mode which probably gives the linearized description of the way bubbles are initiated in hot beds.

Thus, it appears that the effect of large compressibility and heat effects on gas fluidized beds with $R \gg 1$ does not result in any more unstable modes of oscillation that those found in incompressible beds. Although further modes of oscillation are introduced they appear to be stable.

Although the general case, when δ may have any value, has not been investigated in detail it is most unlikely that any further unstable modes will result. In (75) it is understood that Z_0 nowhere becomes small. It might be thought that high gas flow rates would produce this. However, since there is an unstable mode as soon as the bed is fluidized this will result in bubble formation and any excess gas over the incipient fluidization requirement will move up through the bed in the form of bubbles. Experimentally this appears to be the case. In fact increasing the gas flow increases the number of bubbles. If the velocity is high enough transport might take place. Certainly the effect of high gas flow rate does not in general markedly reduce Z_0 uniformly.

In view of the similarity of stability effects found in the above beds as compared with the effects found in the incompressible beds in § 4, it is most probable that similar surface wave effects (that is all surface waves are stable) would be found. The algebra involved in the study of surface waves would be complicated and from physical analogy seems unnecessary. Experimentally surface waves are stable and do not appear to differ from those found in the beds discussed in §§ 4, 5 and so will not be further considered.

7. Internal stability of incompressible centrifugal beds with $R \gg 1$

In view of the practical and effective use of gas-fluidized beds as heat exchangers in the application to a nuclear power source it has been suggested that the use of a centrifuge might allow even greater efficiency to be obtained and might perhaps result in a stable fluidized state. In this section we consider the case of an infinitely long cylindrical bed rotating about its axis with angular velocity Ω which is large enough so that the centrifugal force is large compared with gravity. It is shown below that such beds are unstable to small internal disturbances as in the above cases. It is, however, potentially a more efficient heat exchanger because of the higher gas velocities possible. The interesting case of a finite length cylindrical bed is not studied here.

Let r, θ, z be cylindrical co-ordinates with respective unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$, $\mathbf{k}$ and with rotation $\mathbf{\Omega} = \Omega\mathbf{k}$. We shall consider incompressible beds and the equations, including rotation, are (assuming $\Omega^2 r \gg g$) from (1), (2), (9) and (14)

$$\operatorname{div} Z\mathbf{v}_s = -\partial Z/\partial t, \tag{77}$$

$$\operatorname{div}(1-Z)\mathbf{v}_f = \partial Z/\partial t, \tag{78}$$

$$\rho_f(1-Z)[d\mathbf{v}_f/dt_f + 2\mathbf{\Omega}\times\mathbf{v}_f - \Omega^2 r\mathbf{i}_r] + \rho_s Z[d\mathbf{v}_s/dt_s + 2\mathbf{\Omega}\times\mathbf{v}_s - \Omega^2 r\mathbf{i}_r]$$
$$= -\operatorname{grad} p + (1-Z)\operatorname{div}\boldsymbol{\sigma}_f + Z\operatorname{div}\boldsymbol{\sigma}_s, \tag{79}$$

$$\rho_f(1-Z)[d\mathbf{v}_f/dt_f + 2\mathbf{\Omega}\times\mathbf{v}_f - \Omega^2 r\mathbf{i}_r]$$
$$= -\operatorname{grad} p - D(Z)(\mathbf{v}_f - \mathbf{v}_s) + (1-Z)\operatorname{div}\boldsymbol{\sigma}_f. \tag{80}$$

In (79) we have omitted the gravity term and in (80) the additional mass term. At first sight we look for a steady state in which

$$\mathbf{v}_f = (v_{f_1}, v_{f_2}, 0), \quad \mathbf{v}_s = (0, v_{s_2}, 0), \quad \mathbf{\Omega} = (0,0,\Omega)$$

(that is anticipating $\mathbf{i}_\theta$-components because of $\mathbf{\Omega}$) with all quantities functions of r only. The basic fluidizing velocity comes from the v_{f_1} which is directed inwards to the axis and so is negative. In gas-fluidized beds with $R \gg 1$, the steady-state first-order r-component of (79) may be taken as

$$\rho_s Z[v_{s_2}^2/r + 2\Omega v_{s_2} + \Omega^2 r] = dp/dr, \tag{81}$$

since $[\operatorname{div}\boldsymbol{\sigma}_s]_r = 0$ because $v_{s_1} = 0$ and $[\operatorname{div}\boldsymbol{\sigma}_f]_r$ is small (because μ and ρ_f are small for the gas) compared with the solids momentum. The steady-state θ-component of (79) gives

$$[Z\operatorname{div}\boldsymbol{\sigma}_s]_\theta = \rho_f(1-Z)[v_{f_1}dv_{f_2}/dr + v_{f_1}v_f/r] - (1-Z)[\operatorname{div}\boldsymbol{\sigma}_f]_\theta, \tag{82}$$

and so $[Z\operatorname{div}\boldsymbol{\sigma}_s]_\theta$ is of second order of smallness $(O(\rho_f, u))$. The first-order r-component of (80) is

$$dp/dr = -D(Z)v_{f_1}, \tag{83}$$

and the first-order θ-component of (80) gives

$$v_{f_2} = v_{s_2}. \tag{84}$$

The last equation and (82) imply that v_{s_2} and therefore v_{f_2} are second-order small quantities. In view of the above the first-order steady-state form, instead of the above, must be taken as

$$\mathbf{v}_f = (v_{f_1}, 0, 0) = (v, 0, 0),$$
$$\mathbf{v}_s = 0, \qquad \mathbf{\Omega} = (0,0,\Omega),$$

where from (78), (79), (80), (81), (83) and (84)

$$\left.\begin{aligned} r\rho_f(1-Z)v &= \text{const.} = -W \quad (W>0), \\ dp/dr &= \rho_s Z\Omega^2 r, \\ dp/dr &= -D(Z)v, \end{aligned}\right\} \tag{85}$$

with solutions

$$\left.\begin{aligned} p &= \text{const.} - \rho_s\Sigma(\Omega r) + \tfrac{1}{2}\rho_s(\Omega r)^2, \\ Z &= 1 - \Sigma/\Omega r, \\ v &= \text{const.} - W\Omega/\rho_f\Sigma, \quad \Sigma^2 = \mu a H W/m\rho_f, \end{aligned}\right\} \tag{86}$$

where as in § 6 we have approximated for $D(Z)$ from (12) by taking $G = 1 = Z_s$.

In the above we are considering the bed to be an annulus with $r_0 \leqslant r \leqslant r_i$ (the 'depth' of the bed is $r_i - r_0$). Since the steady state has only one component of

$\mathbf{v}_f$ in the $\mathbf{i}_r$ direction we can approximate the system by an equivalent Cartesian system if we put

$$V = -W, \quad y = r_i - r, \quad \mathbf{j} = -\mathbf{i}_r, \quad \mathbf{i} = \mathbf{i}_\theta, \tag{87}$$

with a steady state

$$\left.\begin{aligned} &\mathbf{v}_s \equiv 0, \quad \mathbf{v}_f = v\mathbf{j} = \text{const.}, \\ &Z = 1 - \Sigma/\Omega(r_i - y), \\ &p = (p)_0 = p_0 - \rho_s \Sigma\Omega(r_i - y) + \tfrac{1}{2}\rho_s \Omega^2 (r_i - y)^2, \end{aligned}\right\} \tag{88}$$

where p_0 is the pressure at the 'bottom' of the bed. The system thus consists of the steady state (88) with the first-order equations, from (77) to (80), becoming (77), (78) and, with $G = 1 = Z_s$ in $D(Z)$,

$$\left.\begin{aligned} \rho_s Z(d\mathbf{v}_s/dt_s + 2\Omega\mathbf{k}\times\mathbf{v}_s) &= -\rho_s Z\Omega^2(r_i - y)\,\mathbf{j} - \operatorname{grad} p + Z \operatorname{div} \boldsymbol{\sigma}_s, \\ \operatorname{grad} p &= -[\mu A H \rho_s Z/m(1-Z)]\,(\mathbf{v}_f - \mathbf{v}_s). \end{aligned}\right\} \tag{89}$$

The propagation of small internal disturbances in such a system is a combination of that studied in § 4 except that there is a modified 'gravity' term and the steady state is not one of constant components, the latter being comparable to that in § 6, and must be similarly treated. We thus look for solutions of the form

$$\left.\begin{aligned} &\mathbf{v}_f = v_0[\mathbf{j} + \mathbf{v}_f' E], \quad \mathbf{v}_s = v_0 \mathbf{v}_s' E, \\ &Z = Z_0(1 + Z'E), \quad p = (p)_0 + p_0 p' E, \\ &E = \exp[i\delta(x - ct) + \lambda(y - y_0)], \end{aligned}\right\} \tag{90}$$

where the subscript zero, except on the co-ordinates and p_0, denote the steady-state solutions given by (88) and $y_0 = r_i - r_0$. As in § 6 we adopt suitable average constant values for the variable co-efficients which arise from the fact that neither Z_0 nor the 'gravity' are constant after manipulation in the equations.

The linearized equations from equations (77), (78) and (89) give

$$\left.\begin{aligned} z v_{f_2}' + i v_{f_1}' + [D(i\omega N - z) - K_5]\,Z' &= 0, \\ (z + K_4)\,v_{s_2}' + i v_{s_1}' - i\omega N Z' &= 0, \\ v_{s_2}'[\{I - i\omega\}\mathbf{j} - \{J + 2\nu\}\mathbf{i}] + v_{s_1}'[\{2\nu - K\}\mathbf{j} + \{L - i\omega\}\mathbf{i}] & \\ + \delta(r_i - y)\,\nu^2 N Z'\mathbf{j} + [z\mathbf{j} + i\mathbf{i}]\,p'P/RD &= 0, \\ N\mathbf{v}_f' - N\mathbf{v}_s' + \{N/(1 - Z_0)\}\,Z'\mathbf{j} + [z\mathbf{j} + i\mathbf{i}]\,p'P/RD &= 0, \end{aligned}\right\} \tag{91}$$

where here

$$\left.\begin{aligned} &\nu = \Omega/(\delta g)^{\frac{1}{2}}, \quad K_4 = (1/\delta Z_0)\,dZ_0/dy, \quad K_5 = (1/\delta)\,dD/dy, \\ &K_6 = (1/\delta)\,d[D(BD^2 + \tfrac{4}{3}A)]/dy, \quad K_8 = (A/\delta)\,dD/dy = AK_5, \\ &I = \Pi_2 - (\Pi_1 + \Pi_2)z^2 - z\theta_2 K_6, \quad J = i[\Pi_1 z + \theta_2 K_8], \\ &K = i[\Pi_1 z + \theta_2(K_6 - 2K_8)], \qquad D = Z_0/(1 - Z_0), \\ &L = (\Pi_1 + \Pi_2) - \Pi_2 z^2 - z\theta_2 K_8, \end{aligned}\right\} \tag{92}$$

with Z_0 from (88). The dispersion relation for (91), where the necessary appropriate averages are assumed to be taken, becomes

$$\begin{aligned}\omega^3(z^2-1)+i\omega^2[(I+L)(z^2-1)+N(D+1)\{z(z+K_4)-1\}-NzK_4]\\ +\omega[(1-z^2)\{IL+(2\nu-K)(2\nu+J)-\delta(r_i-y)\nu^2(z+K_4)\}\\ +\{(D+\{1/(1-Z_0)\})z+K_5\}\{1-z(z+K_4)\}-iK_4\{N(2\nu-K)+iNzL\}\\ -iN(D+1)\{iI-K_4(2\nu-K)+z(J+K)-iz(z+K_4)L\}]\\ +[(1-z^2)\delta(r_i-y)\nu^2\{z\nu+J-i(z+K_4)L\}-K_4 i\delta(r_i-y)\nu^2N\\ +\{(D+\{1/(1-Z_0)\})z+K_5\}\{iI-K_4(2\nu-K)+z(J+K)-iz(z+K_4)L\}]=0.\end{aligned} \tag{93}$$

There are three possible modes of oscillation as in the $R \gg 1$ case in § 4. The (r_i-y) in (93) and (94) is some average value. In view of the similarity between this case and those in §§ 4 and 6 we shall consider only the $\delta \gg 1$ case, in which event (93) becomes

$$(z^2-1)[i\omega+\Pi_2(z^2-1)][\omega^2+i\omega(\Pi_1+\Pi_2)(1-z^2)+\delta(r_i-y)\nu^2 z]=0, \tag{94}$$

assuming that Ω is large enough and $O(\Pi_2)$ or $O(\delta^{\frac{1}{2}}) > O(\nu)$. This last condition for some δ merely allows the factorization in (94) and does not alter the over-all character. Clearly from (94) one stable mode is the same as ω_{12} in (34). However, an unstable mode is evident in the last bracket of (94). Note that this unstable mode is that which compares with the neutrally stable mode which results when $\delta \gg 1$ in ω_{01} in (34). Thus in the case of centrifugal beds if Ω is large enough so that gravity may be neglected there is an unstable mode which in the case of small wavelengths appears to give a growth rate larger than that found in incompressible gas fluidized beds. However, since no more unstable modes appear then in the non-rotating beds the use of centrifugal beds are potentially more efficient because of the higher v allowed than in ordinary beds, for the use envisaged in the introduction to this section.

In view of the similarity between the beds discussed here and in § 4 it is most unlikely that any new surface phenomena would result. The method of § 5 is applicable.

8. Electromagnetic incompressible beds with $R \gg 1$

Fluidized beds in which the particles are electrically conducting may have their flow characteristics changed by the application of electromagnetic fields. This could be of importance in connexion with bubble motion in a bed with $R \gg 1$. In magnetohydrodynamics it is known that an aligned magnetic field can inhibit the onset of instability in a conducting fluid under certain circumstances. In this section we shall thus consider the internal stability of a bed in which the particulate phase is considered an infinitely conducting medium and which is subjected to a magnetic field aligned with the fluidizing velocity. We envisage a circular bed inside a solenoid: no external electric field is present. Consistent first-order equations for $R \gg 1$ and ρ_f constant are (1), (2) and

$$\rho_s Z\, d\mathbf{v}_s/dt_s = -\rho_s Zg\mathbf{j} - \operatorname{grad} p + Z \operatorname{div} \boldsymbol{\sigma}_s + \nu(\operatorname{curl}\mathbf{H})\times\mathbf{H}, \tag{95}$$

$$\operatorname{grad} p = -D(Z)(\mathbf{v}_f - \mathbf{v}_s), \tag{96}$$

$$\partial \mathbf{H}/\partial t + \operatorname{curl}(\mathbf{H} \times \mathbf{v}_s) = 0, \tag{97}$$

$$\mathbf{E} + \nu \mathbf{v}_s \times \mathbf{H} = 0, \tag{98}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field and ν is now the magnetic permeability. $\mathbf{H}$ must be such that $\operatorname{div} \mathbf{H} = 0$. Equations (1), (2) (with ρ_f constant), (95), (96), (97) are a closed set for $\mathbf{v}_s$, $\mathbf{v}_f$, Z, p, $\mathbf{H}$.

As above we consider perturbations of the uniform steady state

$$\mathbf{v}_s = 0, \quad \mathbf{v}_f = v_0 \mathbf{j}, \quad \mathbf{H} = H_0 \mathbf{j}, \quad \mathbf{E} = 0, \quad p = (p)_0 = p_0 - \rho_s g y, \tag{99}$$

and, as in § 4 since multipliers are constants, we look for exponential solutions involving $\exp[i\delta(x-ct) + \lambda(y-y_0)]$. In the same way as in § 4, we finally obtain a corresponding dispersion relation

$$\begin{aligned} &\omega(z^2-1)[i\omega + \Pi_2(z^2-1)][\omega^2 + i\omega\{N(D+1) + (\Pi_1+\Pi_2)(1-z^2)\} - z(D+\mathscr{D})] \\ &\quad - (i\beta/N^2)(z^2-1)[\omega^2(1-z^2) \\ &\quad + i\omega\{-z^2N(D+1) + \Pi_2(1-z^2)^2 - z^2\Pi_1(1-z^2)\} \\ &\quad + z\{1+z^2(D+\mathscr{D})\}] = 0, \end{aligned} \tag{100}$$

where $\beta = \nu H_0^2/\rho_s Z_0 v_0^2$, a dimensionless measure of the magnetic pressure, and Π_1, Π_2, D, $\mathscr{D}$, N are given by relations (29).

Clearly for small β (100) reduces to (34) which has an unstable mode. If $\beta \gg 1$ then (100) has an approximate solution

$$\omega^2(1-z^2) + i\omega\{-z^2N(D+1) + \Pi_2(1-z^2)^2 - z^2\Pi_1(1-z^2)\} + z\{1+z^2(D+\mathscr{D})\} = 0,$$

which clearly has an unstable mode of oscillation when z is a given imaginary number.

It appears, therefore, that nothing is to be gained by applying a magnetic field to a bed, with electrically conducting particles, from the point of view of inhibiting instability. Use of this technique, however, would probably reduce the velocity of rise of bubbles, which are a result of the instability, in such a bed.

9. Conclusions

The equations derived in §§ 2 and 3 of this paper show that fluidized beds are unstable when subjected to a small internal disturbance. It is this instability which probably gives the linearized description of the manner in which bubbles start in beds with $R \gg 1$, and the manner in which turbulence starts in beds with $R < O(10)$. The expressions (or quadratic equations in ω) for the initial growth and velocity of propagation of small disturbances derived in § 4 provide a means of classifying incompressible fluidized systems over a wide range of density ratio of the two phases. The expressions for ω derived in §§ 6, 7 and 8 provide a similar means of classification for hot, centrifugal and electromagnetic beds. In the latter the computation would be considerable.

In § 5 a method is given for the treatment of surface waves in fluidized beds. It is shown conclusively that these waves are all stable in the case of incompressible beds with $R \gg 1$. In view of the similarity between these beds and the others discussed, it is probable that surface waves are stable for all R. The reason for the rapid attenuation of surface waves is shown to be a direct consequence of a particulate viscous stress tensor. The specific form of this tensor is not actually required but the inclusion of a shear *and* bulk viscosity is essential. The form of the tensor is given in (8) and derived in the appendix.

In industry gas fluidized beds with $R \gg 1$ have been most useful. Approximate equations in this case are given by (15) for incompressible beds, (51) for hot beds, and the first two equations of (15) together with (89) for centrifugal incompressible beds. The appropriate combination of the last two will provide approximate equations for hot centrifugal beds.

Although fluidized beds are shown to be unstable there is nothing in the above results which can give any conclusive reason as to why bubbles appear in beds with $R > O(10)$ and not in beds, with $R < O(10)$. Non-linear effects will have to be included in attempting an answer to this question. In the case of centrifugal beds if the bed is of finite length end effects are very important. These are not included in any of the above work. Clearly practical implications of any such effects are of importance and interest.

In a later paper it will be assumed that a bubble has been formed and the resulting bed is stable apart from other bubbles, as observed, and the respective motions of the bubble and two phases studied in detail and compared with experiment.

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Appendix

The form of the solids stress tensor $\boldsymbol{\sigma}_s$ is given by (7). The nature of the shear viscosity μ_s and the bulk viscosity ζ_s is effectively that suggested by Carrier & Cashwell (1956) and is given below.

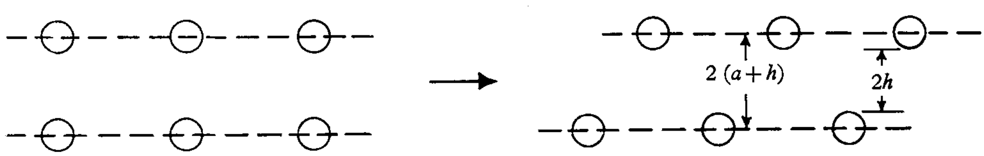

FIGURE 2. Particle motion for shear viscosity.

If a particle configuration before and after relative shear motion is as illustrated in figure 2 then we approximate to the ratio below as follows:

Fluid distortion/Particle configuration distortion

$$\approx O(\{h+a\}/h) = O(1+a/h).$$

We thus get $$\mu_s \approx O(\mu\{a/h\}) \approx O(\mu D_s),$$

since $a/h \approx O(D_s)$ where $D_s = Z/Z_s - Z$. We therefore approximate to μ_s by writing

$$\mu_s = \mu A D_s,$$

where A is a constant, depending on the geometry, of $O(1)$.

For the bulk viscosity ζ_s we consider the separation of two cylinders of radius a lying close together as shown in figure 3. The fluid motion due to the velocity W is taken to be slow enough so that the induced viscous stresses are important. They are, of course, dissipative forces.

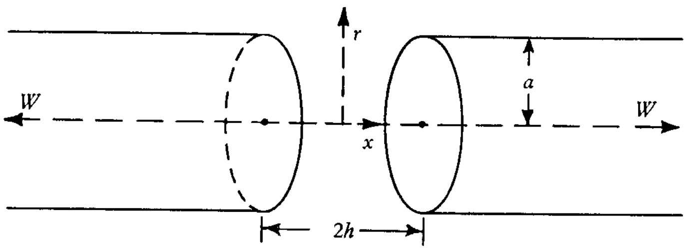

FIGURE 3. Diagram for bulk viscosity.

The radial velocity v, say, is such that

$$v \propto (h^2 - x^2)/h^2$$

$$= v_0(h^2 - x^2)/h^2, \quad \text{say.}$$

Conservation gives $$\pi a^2 \, 2W = \int_{-h}^{h} 2\pi a v \, dx = -\tfrac{8}{3}\pi a v_0 h,$$

which gives $$v_0 = -\tfrac{3}{4} a W/h.$$

At $r = a$, $$v = -\tfrac{3}{4} a W(h^2 - x^2)/h^3,$$

and so conservation is satisfied by

$$v = -\tfrac{3}{4} r W(h^2 - x^2)/h^3.$$

The momentum balance is $$\partial p/\partial r = \mu \, \partial^2 v/\partial x^2,$$

and so $$p(r,x) = p(a,x) - \mu \tfrac{3}{4} W(a^2 - r^2)/h^3,$$

which gives the force F on the particle as $\mu \, 3\pi a^4 W/8h^3$. In the fluid/particle system if the force is considered to be isotropic and W is identified with $\operatorname{div} \mathbf{v}_s$ times the distance between the centres we get a stress tensor contribution of the form

$$\delta_{ij} \mu B(a/h)^3 (\operatorname{div} \mathbf{v}_s) \approx \delta_{ij} \mu B D_s^3 \operatorname{div} \mathbf{v}_s,$$

where B is a constant depending on the geometry but of $O(1)$. We thus take

$$\zeta_s = \mu B D_s^3.$$

The μ_s, ζ_s above are those used in (7) resulting in (8). In most fluidized beds D_s is large and so the particulate bulk viscosity is large compared with the shear viscosity.

Partial list of symbols

A, B, C, G constants $O(1)$

D $= Z_0/(1-Z_0)$

$D(Z) = \mu a H \rho_s Z(1+GD_s)/m$, modification to Stokes drag, equation (12)

D_s $= Z/(Z_s - Z)$

D_{s_0} $= Z_0/(Z_s - Z_0)$

$\mathscr{D}$ $= GD_{s_0}(1+D_{s_0})/(1+GD_{s_0})$

H $=$ average $[6\pi(1+0{\cdot}15\,\mathrm{Re}^{0{\cdot}657})]$

I $= \Pi_2 - (\Pi_1+\Pi_2)(z^2+2zK_3+K_7) - \theta_2 K_6(z+K_3)$, in §6

$= \Pi_2 - (\Pi_1+\Pi_2)z^2 - z\theta_2 K_6$, in §7

J $= i[\Pi_1(z+K_3)+\theta_2 K_8]$, in §6

$= i[\Pi_1 z+\theta_2 K_8]$, in §7

K $= i[\Pi_1(z+K_3)+\theta_2(K_6-2K_8)]$, in §6

$= i[\Pi_1 z+\theta_2(K_6-2K_8)]$, in §7

$K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9, K_{10}$ defined by (60) for §6

K_4, K_5, K_6, K_8 defined by (92) for §7

L $= (\Pi_1+\Pi_2) - \Pi_2(z^2+2zK_3+K_7) - \theta_2 K_8(z+K_3)$, in §6

$= (\Pi_1+\Pi_2) - \Pi_2 z^2 - z\theta_2 K_8$, in §7

L_1, L_2 functions of ω, z defined by (31), (32)

$M(y) = \int_0^y \rho_s Z\,dy$

N $= (g/\delta v_0^2)^{\frac{1}{2}}$

P $= (\delta/g)^{\frac{1}{2}} p_0/V$

R $= \rho_s/\rho_f$

U $= RD/(1+RD)$

V $= \rho_f(1-Z)v$, mass flux in §6, $= -W$, in §7, equation (87)

W $= N + \{C/N(R-1)\}(z-i\omega N)$, equation (41);

$= r\rho_f(1-Z)v$, in §7, equation (57)

Y $= (1/RDN)(z-i\omega N) - (\theta_2/D)(z^2-1)$, equation (41)

z $= \lambda/\delta$, dimensionless wave number

Z $= n\tau = d\xi/d\tau$, fraction of particles in unit volume

α $= R_s \Gamma/c_p V T_f(0)$, equation (54)

β $= T_f(0)\mathscr{R}\mu(0)\,aH/mgp(0)$, equation (54); in §8, $= \nu H_0^2/\rho_s Z_0 v_0^2$

γ $= c_p/c_v$; in §6, equation (54), $= \Gamma g/p(0)$

Γ $= M(y_0) = \int_0^{y_0} \rho_s Z\,dy$

η $= -i\omega$

$\theta_1 \quad = (\frac{1}{3} + \zeta/\mu)\,\theta_2$

$\theta_2 \quad = (\mu^2\delta^3/g\rho_s^2)^{\frac{1}{3}}$

$\nu \quad = \Omega/(\delta g)^{\frac{1}{2}}$ in § 7, equation (92); in § 8, magnetic permeability

$\xi \quad = M/\Gamma$

$\Pi_1 \quad = D_{s_0}(\frac{1}{3}A + BD_{s_0}^2)\,\theta_2$

$\Pi_2 \quad = AD_{s_0}\theta_2$

$\Sigma \quad = [\mu aHW/m\rho_f]^{\frac{1}{2}}$ in § 7

$\phi \quad = T_{s_0}/T_{f_0}$

$\omega \quad = \delta c/(\delta g)^{\frac{1}{2}}$

$\Omega \quad = \delta T_{f_0}/(dT_{f_0}/dy)$ in § 6, equation (60); in § 7, magnitude of rotation

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